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REAL TORIC MANIFOLDS AND SHELLABLE POSETS ARISING FROM GRAPHS

SEONJEONG PARK (OCAMI)

The purpose of this paper is to introduce joint work with Boram Park [12] from a toric topological view.

1. Motivation

Throughout this paper, a graph permits multiple edges but not a loop, and a simple graph means a graph having neither multiple edges nor a loop.

A *toric variety* of complex dimension n is a normal algebraic variety over $\mathbb{C}$ with an effective action of $(\mathbb{C}^*)^n$ having an open dense orbit. A *real toric manifold* is the subset consisting of points with real coordinates of a complete smooth toric variety. The fundamental theorem of toric geometry says that there is a one-to-one correspondence between the class of toric varieties of complex dimension n and the class of fans in $\mathbb{R}^n$. In particular, for a complete smooth toric variety X, the fan Σ_X is complete and smooth. Furthermore, if a smooth toric variety X is projective, then Σ_X can be realized as the normal fan of a Delzant polytope in $\mathbb{R}^n$, where a *Delzant polytope* is a simple convex polytope such that the n primitive vectors (outwardly) normal to the facets meeting at each vertex form a $\mathbb{Z}$-basis. Note that the normal fan of a Delzant polytope is a complete non-singular fan and hence it defines a complete smooth toric variety and a real toric manifold as well.

It is known by Danilov [10] and Jurkiewicz [11] that the (integral) Betti numbers of a complete smooth toric variety X vanish in odd degrees and the $2i$th Betti number of X is equal to h_i, where $(h_0, \dots, h_n)$ is the h-vector of Σ_X. Note that the ith mod 2 Betti number of a real toric manifold $X_\mathbb{R}$ is also equal to h_i. However, unlike toric varieties, only little is known about the cohomology of real toric manifolds. In [14] and [15], Suciu and Trevisan have found a formula for the rational cohomology groups of a real toric manifold, see also [8].

Recently, the rational Betti numbers of some interesting family of real toric manifolds, arising from graphs, have been formulated in terms of some posets determined by a graph by using the Suciu-Trevisan formula, see [7,9]. For a graph G, a simple polytope P_G was introduced in [5,6] as iterated truncations of the product of standard simplices.[1] Furthermore, P_G can be realized as a Delzant polytope canonically, see [7,9] for more details. Hence there is a real toric manifold M_G corresponding to a graph G.

Theorem 1.1 ([9]). *The ith rational Betti number of the real toric manifold M_G is*

$$\beta^i(M_G) = \sum_{\substack{H: \text{PI-graph} \\ \text{of } G}} \sum_{A \in \mathcal{A}(H)} \tilde{\beta}^{i-1}(\Delta(\overline{\mathcal{P}^{\text{odd}}_{H,A}})),$$

where $\Delta(\overline{\mathcal{P}^{\text{odd}}_{H,A}})$ is the ordered complex of the proper part of the poset $\mathcal{P}^{\text{odd}}_{H,A}$.

In Section 2, we will define a PI-graph H of G, an admissible collection $\mathcal{A}(H)$ of H, the poset $\mathcal{P}^{\text{odd}}_{H,A}$, and the poset $\mathcal{P}^{\text{even}}_{H,A}$ satisfying that $\tilde{H}^i(\Delta(\overline{\mathcal{P}^{\text{odd}}_{H,A}})) \cong \tilde{H}_{\dim(P_H)-i-2}(\Delta(\overline{\mathcal{P}^{\text{even}}_{H,A}}))$.

[1] In [5], G is assumed to be simple and P_G is called a graph associahedron, but in [6], G is not necessarily simple and P_G is called a pseudograph associahedron. Note that G having a loop defines an unbounded polyhedron.

A simplicial complex is *shellable* if its facets can be arranged in linear order $F_1, F_2, \ldots, F_t$ in such a way that the subcomplex $(\sum_{i=1}^{k-1} \overline{F_i}) \cap \overline{F_k}$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \ldots, t$. A bounded[2] poset $\mathcal{P}$ is said to be *shellable* if its order complex $\Delta(\mathcal{P})$ is shellable. It is shown in [3] that for a shellable poset $\mathcal{P}$, the order complex $\Delta(\overline{\mathcal{P}})$ is homotopy equivalent to a wedge of spheres (of various dimensions).

Theorem 1.2 ([7]). *Let H be a simple graph. If each of connected components of H has even number of vertices, then $\mathcal{A}(H) = \{V(H)\}$ and $\mathcal{P}^{\text{even}}_{H,V(H)}$ is pure and shellable; otherwise $\mathcal{A}(H) = \emptyset$. Furthermore,*

$$\beta^i(M_G) = \sum_{\substack{I \subseteq V(G) \\ |I| = 2i}} \mu(\mathcal{P}^{\text{even}}_{G|_I, I}), \tag{1.1}$$

where $G|_I$ is the subgraph of G induced by I and $\mu(\mathcal{P}^{\text{even}}_{G|_I,I})$ is the Möbius invariant of the poset $\mathcal{P}^{\text{even}}_{G|_I,I}$.

For instance, for a simple connected path graph,

$$\mu(\mathcal{P}^{\text{even}}_{P_{2k},[2k]}) = \frac{1}{k+1}\binom{2k}{k} \text{ and } \beta^i(M_{P_n}) = \binom{n}{i} - \binom{n}{i-1} \tag{1.2}$$

for $1 \le i \le \lfloor \frac{n}{2} \rfloor$, where $[2k] = \{1, 2, \ldots, 2k\}$. Note that $\frac{1}{k+1}\binom{2k}{k}$ is known as the kth Catalan number and denoted by C_k. In [7], we can find not only (1.2) but also the explicit formula for the rational Betti numbers of M_G when G is a complete graph, a cycle graph, or a star graph. The rational Betti numbers of M_G for complete multipartite graphs are computed in [13].

When G is a simple graph, every PI-graph of G is an induced subgraph of G, and hence Theorem 1.1 is a generalization of (1.1). But, in general, for a non-simple graph G, our posets $\mathcal{P}^{\text{even}}_{H,C}$ and $\mathcal{P}^{\text{odd}}_{H,C}$ are not necessarily to be pure, and many of them are not shellable.

Question ([9]). For a graph G, let $\mathcal{A}^*(G) = \{(H, A) \mid H$ is a PI-graph of G and $A \in \mathcal{A}(H)\}$. Find all graphs G such that $\mathcal{P}^{\text{even}}_{H,A}$ is shellable for every $(H, A) \in \mathcal{A}^*(G)$.

In [12], we answer the question above and give an explicit formula for the rational Betti numbers of the real toric manifolds corresponding to some path graphs having multiple edges.

2. Preliminaries

In this section, we introduce some properties of the polytope P_G, and prepare some notions and basic facts about a poset and its shellability.

Let $G = (V, E)$ be a graph. An edge $e \in E$ is *multiple* if there exists an edge $e'(\neq e)$ in E such that e and e' have the same pair of endpoints. A *bundle* of G is a maximal set of multiple edges which have the same pair of endpoints.[3] A subgraph H of G is an *induced* (respectively, *semi-induced*) subgraph of G if H includes all the edges (respectively, at least one edge) between every pair of vertices in H if such edges exist in G.

Properties of P_G. Let G be a connected graph with vertex set V and bundles $B_1, \ldots, B_k$.

(1) The polytope P_G is constructed from $\Delta^{|V|-1} \times \Delta^{|B_1|-1} \times \cdots \times \Delta^{|B_k|-1}$ by truncating the faces corresponding to the proper connected semi-induced subgraphs of G.[4]

(2) There is a one-to-one correspondence between the facets of P_G and the proper connected semi-induced subgraphs of G.

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[2] A poset $\mathcal{P}$ is said to be *bounded* if it has a unique minimum, denoted by $\hat{0}$, and a unique maximum, denoted by $\hat{1}$. We denote by $\overline{\mathcal{P}} = \mathcal{P} - \{\hat{0}, \hat{1}\}$.

[3] Each bundle of a graph has at least two elements.

[4] The reader can find the detailed construction of P_G in [6, 9].

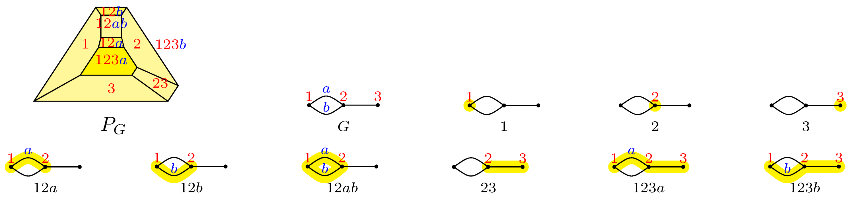


FIGURE 1. The facets of P_G and the proper semi-induced connected subgraphs of G

(3) Two facets F_H and $F_{H'}$ of P_G intersect if and only if H and H' are disjoint and cannot be connected by an edge of G, or one contains the other. See Figure 1.

If $G_1, \ldots, G_\ell$ are connected components of G, then $P_G = P_{G_1} \times \cdots \times P_{G_\ell}$.

A graph H is a *partial underlying graph* of G if H can be obtained from G by replacing some bundles with simple edges, that is, every bundle of H is also a bundle of G. A graph H is a *partial underlying induced* graph (*PI*-graph for short) of G if H is an induced subgraph of some partial underlying graph of G. Now we let $\mathcal{C}_G$ be the set of all the vertices and multiple edges of G. Then every semi-induced subgraph of G can be expressed as a subset of $\mathcal{C}_G$ and for a PI-graph H of G, $\mathcal{C}_H$ is inherited from $\mathcal{C}_G$. See Figures 1 and 2.

For a connected graph H, a subset $A \subset \mathcal{C}_H$ is *admissible* to H if the following hold:

(1) $|A \cap V(H)| \equiv 0 \pmod 2$ and each vertex incident to only simple edges of H is contained in A,
(2) $B \cap A \neq \emptyset$ and $|B \cap A| \equiv 0 \pmod 2$, for each bundle B of H.

For a disconnected graph H, $A \subset \mathcal{C}_H$ is *admissible* to H if $\mathcal{C}_{H'} \cap A$ is admissible to H' for each component H' of H. We denote by $\mathcal{A}(H)$ the set of all the admissible collections of H. For each H_i in Figure 2, we have $\mathcal{A}(H_1) = \{1234\}$, $\mathcal{A}(H_2) = \{1234ab, 34ab\}$, and $\mathcal{A}(H_3) = \{1234cd, 1234ce, 1234de, 14cd, 14ce, 14de\}$.

For each $A \in \mathcal{A}(H)$, a semi-induced subgraph I of H is *A-even* (respectively, *A-odd*) if $|I' \cap A|$ is even (respectively, odd) for each component I' of I. Now we define the poset $\mathcal{P}^{\text{even}}_{H,A}$ (respectively, $\mathcal{P}^{\text{odd}}_{H,A}$) by the poset consisting of all A-even (respectively, A-odd) semi-induced subgraphs of H ordered by subgraph containment, including both $\emptyset$ and H. Note that if $\mathcal{A}(H) = \emptyset$ then $\mathcal{P}^{\text{even}}_{H,A}$ and $\mathcal{P}^{\text{odd}}_{H,A}$ are defined to be the null poset, and if $\mathcal{A}(H) \neq \emptyset$ then $\mathcal{P}^{\text{even}}_{H,A}$ and $\mathcal{P}^{\text{odd}}_{H,A}$ are bounded posets. Figure 2 gives examples of $\mathcal{P}^{\text{even}}_{H,A}$.

Note that for a graph H, $\Delta(\overline{\mathcal{P}^{\text{even}}_{H,A}})$ (respectively, $\Delta(\overline{\mathcal{P}^{\text{odd}}_{H,A}})$) is a geometric subdivision of the simplicial complex dual to the union of the facets F_I of the polytope P_H such that $|I \cap A|$ is even (respectively, odd). Hence, from the Alexander duality, we have $\tilde{H}^i(\Delta(\overline{\mathcal{P}^{\text{odd}}_{H,A}})) \cong \tilde{H}_{\dim(P_H)-i-2}(\Delta(\overline{\mathcal{P}^{\text{even}}_{H,A}}))$.

For a bounded poset $\mathcal{P}$, we denote by $\mathcal{ME}(\mathcal{P})$ the set of pairs $(\sigma, x \lessdot y)$ consisting of a maximal chain σ and a cover $x \lessdot y$ along that chain. For $x, y \in \mathcal{P}$ and a maximal chain r of $[\hat{0}, x]$, the closed rooted interval $[x, y]_r$ of $\mathcal{P}$ is a subposet of $\mathcal{P}$ obtained from $[x, y]$ adding the chain r. A *chain-edge labeling* of $\mathcal{P}$ is a map $\lambda\colon \mathcal{ME}(\mathcal{P}) \to \Lambda$, where Λ is some poset satisfying; if two maximal chains coincide along their bottom d covers, then their labels also coincide along those covers. A *chain-lexicographic labeling* (*CL-labeling* for short) of a bounded poset $\mathcal{P}$ is a *chain-edge labeling* such that for each closed rooted interval $[x, y]_r$ of $\mathcal{P}$, there is a unique strictly increasing maximal chain, which lexicographically precedes all other maximal chains of $[x, y]_r$. A poset that admits a CL-labeling is said to be *CL-shellable*. We can easily see that $\mathcal{P}^{\text{even}}_{H_1,1234}$ and $\mathcal{P}^{\text{even}}_{H_2,1234ab}$ are CL-shellable.

Given a CL-labeling $\lambda\colon \mathcal{ME}(\mathcal{P}) \to \Lambda$, a maximal chain $\sigma : x_0 \lessdot x_1 \lessdot \cdots \lessdot x_\ell$ of $\mathcal{P}$ is called a *falling chain* if $\lambda(\sigma, x_{i-1} \lessdot x_i) \geq_\Lambda \lambda(\sigma, x_i \lessdot x_{i+1})$ for every $1 \leq i < \ell$.

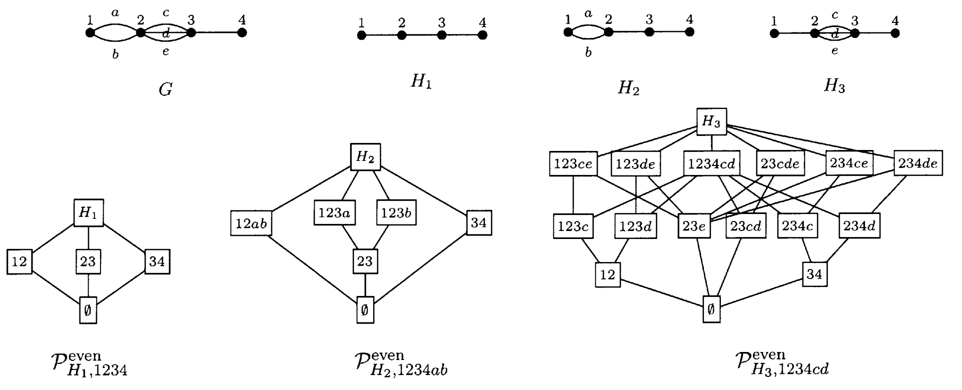


FIGURE 2. Examples for PI-graphs of G and the posets $\mathcal{P}^{\text{even}}_{H,A}$

Theorem 2.1 ([1,3,4]). *The following hold:*

1. *If a bounded poset $\mathcal{P}$ is CL-shellable, then $\Delta(\overline{\mathcal{P}})$ has the homotopy type of a wedge of spheres. Furthermore, for any fixed CL-labeling, the ith reduced Betti number of $\Delta(\overline{\mathcal{P}})$ is equal to the number of falling chains of length $i+2$.*
2. *Every (closed) interval of a shellable (respectively, CL-shellable) poset is shellable (respectively, CL-shellable).*
3. *The product of bounded posets is shellable (respectively, CL-shellable) if and only if each of the posets is shellable (respectively, CL-shellable).*
4. *A bounded poset is pure and totally semimodular, then it is CL-shellable.*

By (1) of Theorem 2.1, both $\Delta(\overline{\mathcal{P}^{\text{even}}_{H_1,1234}})$ and $\Delta(\overline{\mathcal{P}^{\text{even}}_{H_2,1234ab}})$ in Figure 2 have the homotopy type $S^0 \vee S^0$ because they have two falling chains of length 2 for any CL-labelling. Theorem 2.1 shows that $\mathcal{P}^{\text{even}}_{H_3,1234cd}$ is not shellable because the interval $[\emptyset, 1234cd]$ is not shellable.

An alternative approach to CL-shellability, via so-called "recursive atom orderings", was introduced in [2,3].

Definition 2.2. A bounded poset $\mathcal{P}$ *admits a recursive atom ordering* if its length $\ell(\mathcal{P})$ is 1, or $\ell(\mathcal{P}) > 1$ and there is an ordering $\alpha_1, \ldots, \alpha_t$ of the atoms of $\mathcal{P}$ satisfying the following:

1. For all $j = 1, \ldots, t$, the interval $[\alpha_j, \hat{1}]$ admits a recursive atom ordering in which the atoms of $[\alpha_j, \hat{1}]$ that belong to $[\alpha_i, \hat{1}]$ for some $i < j$ come first.
2. For all i, j with $1 \le i < j \le t$, if $\alpha_i, \alpha_j < y$ then there exist an integer k and an atom z of $[\alpha_j, \hat{1}]$ such that $1 \le k < j$ and $\alpha_k < z \le y$.

Theorem 2.3 ([3]). *A bounded poset admits a recursive atom ordering if and only if it is CL-shellable.*

3. MAIN RESULT AND ITS APPLICATION

In this section, we introduce the main result in [12] and give the formula for the rational Betti numbers of $M_{\tilde{P}_{n,2}}$ as an application, where $\tilde{P}_{n,2}$ is a graph in Figure 3.

Let $\mathcal{G}$ be the collection of graphs whose connected components are simple or belong to the list in Figure 3.

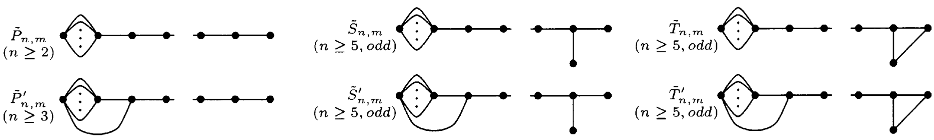


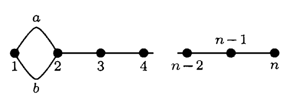
FIGURE 3. Non-simple connected graphs with n vertices and m multiple edges ($m \geq 2$)

Theorem 3.1 (Main result in [12]). *Let G be a graph. Then $\mathcal{P}^{\text{even}}_{H,A}$ is CL-shellable for every $(H, A) \in \mathcal{A}^*(G)$ if and only if G belongs to $\mathcal{G}$.*

Sketch of proof. The proof of 'only if' part relies on (2) of Theorem 2.1; if a graph G is not in $\mathcal{G}$, then we can always find a pair $(H, A) \in \mathcal{A}^*(G)$ such that $\mathcal{P}^{\text{even}}_{H,A}$ has a non-shellable interval, see Theorem 4.2 in [12].

The proof of the 'if' part relies on (3)$\sim$(4) of Theorem 2.1 and Theorem 2.3. For a simple connected graph H, if $\mathcal{A}(H) \neq \emptyset$, then $\mathcal{P}^{\text{even}}_{H,V(H)}$ is pure and totally semimodular (see [7]), and hence $\mathcal{P}^{\text{even}}_{H,V(H)}$ is CL-shellable by (4) of Theorem 2.1. For a non-simple connected graph $H \in \mathcal{G}$, $\mathcal{P}^{\text{even}}_{H,A}$ admits a recursive atom ordering for every $A \in \mathcal{A}(H)$ (see Theorem 5.3 in [12]), and hence it is CL-shellable by Theorem 2.3. Since every PI-graph of $G \in \mathcal{G}$ belongs to $\mathcal{G}$, every $G \in \mathcal{G}$ satisfies that $\mathcal{P}^{\text{even}}_{H,A}$ is shellable for every $(H, A) \in \mathcal{A}^*(G)$ by (3) of Theorem 2.1. □

Now we see the rational Betti numbers of the real toric manifold corresponding to $\tilde{P}_{n,2}$ in Figure 3. We give labels $1, \dots, n$ to the vertices from left to right and a, b to the multiple edges as shown below.



Under the recursive atom ordering in Theorem 5.3 in [12], we can compute the number of falling chains of $\mathcal{P}^{\text{even}}_{\tilde{P}_{n,2},A}$, which tells us the homotopy type of $\Delta(\overline{\mathcal{P}^{\text{even}}_{\tilde{P}_{n,2},A}})$ by (1) of Theorem 2.1. Note that

$$\mathcal{A}(\tilde{P}_{n,2}) = \begin{cases} \{A_1 := 12\cdots nab, A_2 := 34\cdots nab\}, & \text{if } n \text{ is even;} \\ \{A_3 := 134\cdots nab, A_4 := 234\cdots nab\}, & \text{if } n \text{ is odd.} \end{cases}$$

Proposition 3.2 (Proposition 6.3 and Table 2 in [12]). *If n is even, then*

$$\Delta(\overline{\mathcal{P}^{\text{even}}_{\tilde{P}_{n,2},A_1}}) \simeq \bigvee_{C_{k-1}} S^{k-3} \text{ and } \Delta(\overline{\mathcal{P}^{\text{even}}_{\tilde{P}_{n,2},A_2}}) \simeq \bigvee_{C_k} S^{k-1}$$

for $k = \frac{n-2}{2}$. If n is odd, then

$$\Delta(\overline{\mathcal{P}^{\text{even}}_{\tilde{P}_{n,2},A_3}}) \text{ is contractible and } \Delta(\overline{\mathcal{P}^{\text{even}}_{\tilde{P}_{n,2},A_4}}) \simeq \bigvee_{C_{k+1}-C_k} S^{k-1}$$

for $k = \frac{n-3}{2}$. Here, C_k is the kth Catalan number.

Note that $\Delta(\overline{\mathcal{P}_{2k,[2k]}})$ is homotopy equivalent to $\bigvee_{C_k} S^{k-2}$. Since each connected component of a PI-graph of $\tilde{P}_{n,2}$ is a simple path graph or $\tilde{P}_{m,2}$ for some $m \leq n$. By using $\tilde{H}^i(\Delta(\overline{\mathcal{P}^{\text{odd}}_{H,A}})) \cong \tilde{H}_{\dim(P_H)-i-2}(\Delta(\overline{\mathcal{P}^{\text{even}}_{H,A}}))$, we can plug Proposition 3.2 into Theorem 1.1 and compute the rational Betti numbers of $M_{\tilde{P}_{n,2}}$.

Proposition 3.3 (Section 6.2 in [12]). *The ith rational Betti number of $M_{\tilde{P}_{n,2}}$ is*

$$\beta^i(M_{\tilde{P}_{n,2}}) \;=\; \beta^i(M_{P_n}) + \sum_{\ell=0}^{i-1}\sum_{m=2}^{n-2} b_m^\ell \beta^{i-\ell-1}(M_{P_{n-m-1}}) + b_{n-1}^{i-1} + b_n^{i-1},$$

where

$$\beta^i(M_{P_n}) = \begin{cases} \binom{n}{i} - \binom{n}{i-1}, & \text{if } 1 \le i \le \lfloor \frac{n}{2} \rfloor; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$b_k^i := \begin{cases} C_{\frac{k}{2}}, & \text{if } i = \frac{k}{2} \text{ or } \frac{k}{2} - 1 \text{ for even } k \\ C_{\frac{k+1}{2}} - C_{\frac{k-1}{2}}, & \text{if } i = \frac{k-1}{2} \text{ for odd } k \\ 0 & \text{otherwise.} \end{cases}$$

For some i, $\beta^i(M_{\tilde{P}_{n,2}})$ can be written in a simple form. For instance, $\beta^1(M_{\tilde{P}_{n,2}}) = n$, $\beta^2(M_{\tilde{P}_{n,2}}) = \binom{n}{2}$, and $\beta^k(M_{\tilde{P}_{2k,2}}) = \beta^{k+1}(M_{\tilde{P}_{2k+1,2}}) = \frac{6k}{k+2}C_k$, which is known as the total number of nonempty subtrees over all binary trees having $k+1$ internal vertices, see [16, A071721].

Remark. It would be interesting if one figures out that the ith rational Betti number $\beta^i(M_G)$ counts other combinatorial objects for every $G \in \mathcal{G}$.

References

[1] A. Björner, *Shellable and Cohen-Macaulay partially ordered sets*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 159–183.

[2] A. Björner and M. L. Wachs, *On lexicographically shellable posets*, Trans. Amer. Math. Soc. **277** (1983), no. 1, 323-341.

[3] A. Björner and M. L. Wachs, *Shellable non pure complexes and posets I*, Trans. Amer. Math. Soc. **348** (1996), no. 4, 1299–1327.

[4] A. Björner and M. L. Wachs, *Nonpure shellable complexes and posets II*, Trans. Amer. Math. Soc. **349** (1997), no. 10, 3945-3975.

[5] M. Carr and S. L. Devadoss, *Coxeter complexes and graph-associahedra*, Topology Appl., **153** (2006), no. 12, 2155–2168.

[6] M. Carr, S. L. Devadoss and S. Forcey, *Pseudograph associahedra*, J. Combin. Theory Ser. A **118** (2011), no. 7, 2035–2055.

[7] S. Choi and H. Park, *A new graph invariant arises in toric topology*, J. Math. Soc. Japan, **67** (2015), no. 2, 699–720.

[8] S. Choi and H. Park, *On the cohomology and their torsion of real toric objects*, Forum Math. **29** (2017), no. 3, 543553.

[9] S. Choi, B. Park and S. Park, *Pseudograph and its associated real toric manifold*, J. Math. Soc. Japan **69** (2017), no. 2, 693–714.

[10] V. I. Danilov, *The geometry of toric varieties*, Uspekhi Mat. Nauk, **33** (1978), no. 2(200), 85–134.

[11] J. Jurkiewicz, *Chow ring of projective nonsingular torus embedding*, Colloq. Math., **43** (1980), no. 2, 261–270.

[12] B. Park and S. Park, *Shellable posets arising from even subgraphs of a graph*, arXiv:1705.06423, 2017.

[13] S. Seo and H. Shin, *Signed a-polynomials of graphs and Poincaré polynomials of real toric manifolds*, Bull. Korean Math. Soc. **52** (2015), no. 2, 467-481.

[14] A. Suciu and A. Trevisan, *Real toric varieties and abelian covers of generalized Davis-Januszkiewicz spaces*, preprint, 2012.

[15] A. Trevisan, *Generalized Davis-Januszkiewicz spaces and their applications in algebra and topology*, Ph.D. thesis, Vrije University Amsterdam, 2012.

[16] The on-line encyclopedia of integer sequences, available at https://oeis.org/.

Department of Mathematics, Osaka City University, Sumiyoshi-ku, Osaka 558-8585, Japan.
E-mail address: `seonjeong1124@gmail.com`